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On the divided differences of the remainder in polynomial interpolation☆

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Abstract

We present formulas for the divided differences of the remainder of the interpolation polynomial that include some recent interesting formulas as special cases.

Keywords: Divided difference; Hermite interpolation polynomial; Newton interpolation formula

1. Introduction

Let $a_0, \ldots, a_n$ be a sequence of fixed nodes in $\mathbb{R}$ or $\mathbb{C}$. Set

$$\omega_v(x) := \prod_{i=0}^{v-1} (x - a_i), \quad v = 0, 1, \ldots, n+1. \tag{1}$$

Then the identity

$$f(x) = \sum_{v=0}^{n} \omega_v(x)[a_0, \ldots, a_v]f + \omega_{n+1}(x)[x, a_0, \ldots, a_n]f \tag{2}$$

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describes the Hermite polynomial interpolant and its error

$$R(x) \coloneqq \omega_{n+1}(x)[x, a_0, \dots, a_n]f.$$

Identity (2) is the main resource for deriving numerical differentiation formulas by differentiating both sides k times for some $k \leqslant n$. Then the remainder for these numerical differentiation formulas is $R^{(k)}(x)$. In [8], the following identity was established:

$$\frac{R^{(k)}(x)}{k!} = \sum_{v=0}^{m-1} \frac{\omega_{n+1}^{(k-v)}(x)}{(k-v)!} \left[\underbrace{x, \dots, x}_{v+1}, a_0, \dots, a_n\right] f + \sum_{v=m}^{k} (x - a_{n+m-v}) \frac{\omega_{n+m-v}^{(k-v)}(x)}{(k-v)!} \left[\underbrace{x, \dots, x}_{v+1}, a_0, \dots, a_{n+m-v}\right] f \tag{3_m}$$

for $m \leqslant k$. In particular, (3_0) is the well-known formula given in [4] (for the special case $k = n$) and [2]. In general, formula (3_0) suffices. However, when $\omega_{n+1}^{(k)}(x) = 0$ while $\omega_{n+1}^{(k-1)}(x) \neq 0$, only (3_m) with $m > 0$ can make explicit the so-called super convergence phenomenon, the appropriate convergence order of the numerical differentiation formula. For example, let $a_0, \dots, a_n$ be $n+1$ equally spaced points in $\mathbb{R}$ with even integer n. By (3_0), the remainder of the second-order numerical differentiation formula at nodes a_i with $i \neq n/2$ gives

$$R''(a_i) = (-1)^{(n-i)} 2 \frac{i!(n-i)!}{(n+1)!} \left(\sum_{v=1}^{i} \frac{1}{v} - \sum_{v=1}^{n-i} \frac{1}{v} \right) h^{n-1} f^{(n+1)}(\xi),$$

where $h = a_1 - a_0$, $a_0 \leqslant \xi \leqslant a + nh$, and we have assumed that $f \in C^{n+1}[a_0, a_n + nh]$.

However, for $i = n/2$, we can use (3_1) to get

$$R''(a_i) = (-1)^i 2 \frac{(i!)^2}{(n+2)!} h^n f^{(n+2)}(\xi)$$

if $f \in C^{n+2}[a_0, a_0 + nh]$ (cf. [8,9]). This higher-order numerical differentiation formula is fundamental in the finite difference method for differential equations (cf. [5, Section 2, Chapter 3]).

For the convenience of dealing with less differentiable functions, Wang and Yang [13] proved the following formula: for any integer m satisfying $k \leqslant m \leqslant n$,

$$\frac{R^{(k)}(x)}{k!} = \sum_{v=0}^{k} (x - a_{m-v}) \omega_{m-v}^{(k-v)}(x) \left[\underbrace{x, \dots, x}_{v+1}, a_0, \dots, a_{m-v}\right] f - \sum_{v=m+1}^{n} \omega_v^{(k)}(x)[a_0, \dots, a_v] f. \tag{4_m}$$

Using (4_{m}), for $f \in C^m[\min\{a_i\}, \max\{a_i\}]$ with $k \leqslant m \leqslant n$, we have

$$||R^{(k)}||_\infty \leqslant \frac{2^{n-m+1} n^k}{m!} \frac{(\bar{a}_n - \bar{a}_0)^{n-k}}{\prod_{j=m+1}^n h_j} \omega(f^{(m)}, h_{m+1}),$$

where $h_j = \min\{\bar{a}_{i+j} - \bar{a}_i, 0 \leqslant i, i+j \leqslant n\}$ and $\bar{a}_0 \leqslant \bar{a}_1 \leqslant \cdots \leqslant \bar{a}_n$ is a rearrangement of $a_0, \ldots, a_n$ and $\omega(f^{(m)}, \cdot)$ denotes the modulus of continuity of $f^{(m)}$. Both formulas (3_{m}) and (4_{m}) have their applications in spline analysis (cf. [14]). They also have useful applications in the study of the convergence of iterative methods (cf. [7]). Wang and Lai [6] and [10] generalized these formulas to the multivariate setting.

Recently, Floater (cf. (2.2) in [3]) and de Boor (cf. (3) in [1]) gave the following formula:

$$\begin{aligned}[x_0, \ldots, x_k]R = \sum_{v=0}^{k} & (x_{k-v} - a_{n-v})([x_0, \ldots, x_{k-v}]\omega_{n-v}) \\ & \times [a_0, \ldots, a_{n-v}, x_{k-v}, \ldots, x_k]f \end{aligned} \tag{5}$$

(rewritten in the terms used here to permit easy comparison with (3_0) above) for the divided difference $[x_0, \ldots, x_k]R$ of the remainder in Hermite interpolation. Floater's identity is for the special case that the interpolation points are the same, i.e., $a_0 = \cdots = a_n$, while de Boor's identity is for general node sequences. His identity can be viewed as an extension of formula (3_0).

In this paper we further study the remainder $[x_0, \ldots, x_k]R$. As we have seen before that the remainder formula (3_0) for the analysis of the numerical differentiation formula was not adequate, so (5) will also not be enough for the analysis of the remainder of numerical divided difference formula. We shall generalize (5) in the spirit of (3_{m}) and(4_{m}). This is the purpose of the paper.

2. Main results and proofs

The main results in this paper are the following

Proposition 2.1. *Let $a_0, \ldots, a_n$ be a sequence of nodes in $\mathbb{R}$ or in $\mathbb{C}$. Let $x_0, \ldots, x_k$ be another sequence of nodes. Then for any integer m satisfying $0 \leqslant m \leqslant k$,*

$$\begin{aligned} &[x_0, \ldots, x_k]R \\ &\quad = \sum_{v=0}^{m-1} [x_v, \ldots, x_k]\omega_{n+1}[x_0, \ldots, x_v, a_0, \ldots, a_n]f \\ &\qquad + \sum_{v=m}^{k} (x_v - a_{n+m-v})[x_v, \ldots, x_k]\omega_{n+m-v}[x_0, \ldots, x_v, a_0, \ldots, a_{n+m-v}]f \end{aligned} \tag{6_{m}}$$

and for $k \leqslant m \leqslant n$,

$$[x_0, \ldots, x_k]R = \sum_{v=0}^{k} (x_v - a_{m-v})[x_v, \ldots, x_k]\omega_{m-v}[x_0, \ldots, x_v, a_0, \ldots, a_{m-v}]f \\ - \sum_{v=m+1}^{n} [x_0, \ldots, x_k]\omega_v[a_0, \ldots, a_v]f. \quad (7_{\rm m})$$

Proof. By (2), we have

$$\begin{aligned}[x_0, \ldots, x_k]R &= [x_0, \ldots, x_k]f - [a_0, \ldots, a_k]f - \sum_{v=k+1}^{n} [x_0, \ldots, x_k]\omega_v[a_0, \ldots, a_v]f \\ &= \sum_{v=0}^{k} ([x_0, \ldots, x_v, a_0, \ldots, a_{k-v-1}]f - [x_0, \ldots, x_{v-1}, a_0, \ldots, a_{k-v}]f) \\ &\quad - \sum_{v=k+1}^{n} [x_0, \ldots, x_k]\omega_v[a_0, \ldots, a_v]f \\ &= \sum_{v=0}^{k} (x_v - a_{k-v})[x_0, \ldots, x_v, a_0, \ldots, a_{k-v}]f \\ &\quad - \sum_{v=k+1}^{n} [x_0, \ldots, x_k]\omega_v[a_0, \ldots, a_v]f\end{aligned}$$

which is (7_k). For $k \leqslant m < n$, substitution into $(7_{\rm m})$ of the following consequence

$$(x_v - a_{m-v})[x_v, \ldots, x_k]\omega_{m-v} = [x_v, \ldots, x_k]\omega_{m+1-v} - [x_{v+1}, \ldots, x_k]\omega_{m-v} \quad (8)$$

of Leibniz' formula produces (7_{m+1}). Induction implies that $(7_{\rm m})$ holds for all $k \leqslant m \leqslant n$.

Note that (7_n) is the same as (6_0). For $0 \leqslant m < k$, substituting (8), with m replaced by $m+n$, into $(6_{\rm m})$, we get (6_{m+1}). Thus, $(6_{\rm m})$ is valid for all $0 \leqslant m \leqslant k$ by induction. □

Letting $m = 0$ in $(6_{\rm m})$, we get the de Boor formula (5). If $a_0 = \cdots = a_n = x$, (6_0) becomes the Floater formula.

If

$$[x_v, \ldots, x_k]\omega_{n+1} = 0, \quad 0 \leqslant v \leqslant m-1, \quad (9)$$

then $(6_{\rm m})$ gives

$$[x_0, \ldots, x_k]R = \sum_{v=m}^{k} (x_v - a_{n+m-v})[x_v, \ldots, x_k]\omega_{n+m-v} \\ \times [x_0, \ldots, x_v, a_0, \ldots, a_{m+n-v}]f. \quad (10)$$

In this case, the numerical divided difference formula possesses the so-called super convergence property. For example, when $a_0 = \cdots = a_n = x$, it follows from (10) that

$$|[x_0, \ldots, x_k]R| \leqslant Ch^{n+1+m-k} \|f^{(n+m+1)}\|_\infty, \tag{11}$$

where $h = \max\{|x_i - x|, i = 0, \ldots, k\}$. Condition (9) holds, e.g., when ω_{n+1} agrees on $x_0, \ldots, x_k$ with some polynomial of degree at most $k - m$. A further study of these remainder formulas is carried out in [12] and several divided difference formulas with super convergence were obtained there.

When $x_0 = \cdots = x_k = x$, (6_m) and (7_m) are (3_m) and (4_m), respectively.

Finally we point out that the results in Proposition 2.1 can be generalized to the multivariate setting. The details can be found in an extended version of this paper (cf. [11]).

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